Chapter 1

Introduction to Mechanical Vibrations

Vibration is the motion of a particle or a body or system of connected bodies displaced from a position of equilibrium. Most vibrations are undesirable in machines and structures because they produce increased stresses, energy losses, cause added wear, increase bearing loads, induce fatigue, create passenger discomfort in vehicles, and absorb energy from the system. Rotating machine parts need careful balancing in order to prevent damage from vibrations.

Vibration occurs when a system is displaced from a position of stable equilibrium. The system tends to return to this equilibrium position under the action of restoring forces (such as the elastic forces, as for a mass attached to a spring, or gravitational forces, as for a simple pendulum). The system keeps moving back and forth across its position of equilibrium. A *system* is a combination of elements intended to act together to accomplish an objective. For example, an automobile is a system whose elements are the wheels, suspension, car body, and so forth. A *static* element is one whose output at any given time depends only on the input at that time while a *dynamic* element is one whose present output depends on past *inputs*. In the same way we also speak of *static* and *dynamic systems*. A *static system* contains all elements while a *dynamic system* contains at least one dynamic element.

A physical system undergoing a time-varying interchange or dissipation of energy among or within its elementary storage or dissipative devices is said to be in a *dynamic state*. All of the elements in general are called *passive*, *i.e.,* they are incapable of generating net energy. A dynamic system composed of a finite number of storage elements is said to be *lumped* or *discrete*, while a system containing elements, which are dense in physical space, is called *continuous*. The analytical description of the dynamics of the discrete case is a set of ordinary differential equations, while for the continuous case it is a set of partial differential equations. The analytical formation of a dynamic system depends upon the kinematic or geometric constraints and the physical laws governing the behaviour of the system.

1.1 CLASSIFICATION OF VIBRATIONS

Vibrations can be classified into three categories: *free, forced*, and *self-excited*. *Free vibration* of a system is vibration that occurs in the absence of external force. An external force that acts on the system causes forced vibrations. In this case, the exciting force continuously supplies energy to the system. Forced vibrations may be either deterministic or random (see Fig. 1.1). *Self-excited vibrations* are periodic and deterministic oscillations. Under certain conditions, the equilibrium state in such a vibration system becomes unstable, and any disturbance causes the perturbations to grow until some effect limits any further growth. In contrast to forced vibrations, the exciting force is independent of the vibrations and can still persist even when the system is prevented from vibrating.

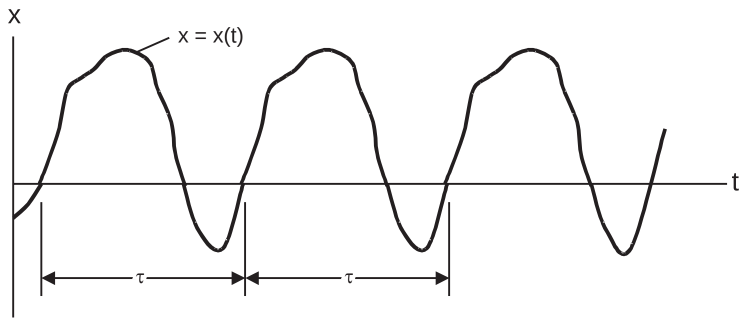

Fig. 1.1(a) A deterministic (periodic) excitation.

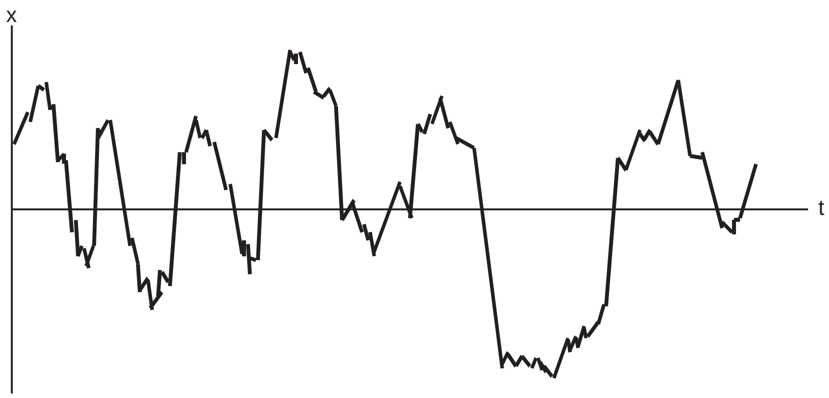

Fig. 1.1(b) Random excitation.

1.2 ELEMENTARY PARTS OF VIBRATING SYSTEMS

In general, a vibrating system consists of a spring (a means for storing potential energy), a mass or inertia (a means for storing kinetic energy), and a damper (a means by which energy is gradually lost) as shown in Fig. 1.2. An undamped vibrating system involves the transfer of its potential energy to kinetic energy and kinetic energy to potential energy, alternatively. In a damped vibrating system, some energy is dissipated in each cycle of vibration and should be replaced by an external source if a steady state of vibration is to be maintained.

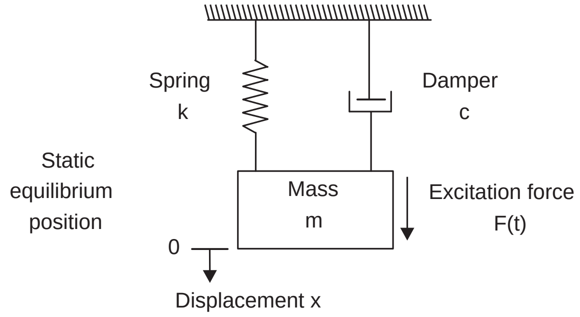

Fig. 1.2 Elementary parts of vibrating systems.

1.3 PERIODIC MOTION

When the motion is repeated in equal intervals of time, it is known as *periodic motion*. Simple harmonic motion is the simplest form of periodic motion. If $x(t)$ represents the displacement of a mass in a vibratory system, the motion can be expressed by the equation

$$x = A \cos \omega t = A \cos 2\pi \frac{t}{\tau}$$

where A is the amplitude of oscillation measured from the equilibrium position of the mass. The repetition time τ is called the *period of the oscillation*, and its reciprocal, $f = \frac{1}{\tau}$, is called the *frequency*. Any periodic motion satisfies the relationship

$$x(t) = x(t + \tau)$$

That is Period $\tau = \frac{2\pi}{\omega}$ s/cycle

Frequency $f = \frac{1}{\tau} = \frac{\omega}{2\pi}$ cycles/s, or Hz

ω is called the *circular frequency* measured in rad/sec.

The velocity and acceleration of a harmonic displacement are also harmonic of the same frequency, but lead the displacement by $\pi/2$ and π radians, respectively. When the acceleration $\ddot{X}$ of a particle with rectilinear motion is always proportional to its displacement from a fixed point on the path and is directed towards the fixed point, the particle is said to have *simple harmonic motion*.

The motion of many vibrating systems in general is not harmonic. In many cases the vibrations are periodic as in the impact force generated by a forging hammer. If $x(t)$ is a periodic function with period τ, its Fourier series representation is given by

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

where $\omega = 2\pi/\tau$ is the fundamental frequency and $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$ are constant coefficients, which are given by:

$$a_0 = \frac{2}{\tau} \int_0^{\tau} x(t)\, dt$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t\, dt$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t\, dt$$

The exponential form of $x(t)$ is given by:

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}$$

The Fourier coefficients c_n can be determined, using

$$c_n = \frac{1}{\tau}\int_0^{\tau} (x)t\, e^{-in\omega t}\, dt$$

The harmonic functions $a_n \cos n\omega t$ or $b_n \sin n\omega t$ are known as the *harmonics of order n* of the periodic function $x(t)$. The harmonic of order n has a period τ/n. These harmonics can be plotted as vertical lines in a diagram of amplitude (a_n and b_n) versus frequency ($n\omega$) and is called *frequency spectrum*.

1.4 DISCRETE AND CONTINUOUS SYSTEMS

Most of the mechanical and structural systems can be described using a finite number of degrees of freedom. However, there are some systems, especially those include continuous elastic members, have an infinite number of degree of freedom. Most mechanical and structural systems have elastic (deformable) elements or components as members and hence have an infinite number of degrees of freedom. Systems which have a finite number of degrees of freedom are known as *discrete* or *lumped parameter systems*, and those systems with an infinite number of degrees of freedom are called *continuous* or *distributed systems*.

1.5 VIBRATION ANALYSIS

The outputs of a vibrating system, in general, depend upon the initial conditions, and external excitations. The vibration analysis of a physical system may be summarised by the four steps:

1. Mathematical Modelling of a Physical System
2. Formulation of Governing Equations
3. Mathematical Solution of the Governing Equations

1. Mathematical modelling of a physical system

The purpose of the mathematical modelling is to determine the existence and nature of the system, its features and aspects, and the physical elements or components involved in the physical system. Necessary assumptions are made to simplify the modelling. Implicit assumptions are used that include:

(*a*) A physical system can be treated as a continuous piece of matter

(*b*) Newton's laws of motion can be applied by assuming that the earth is an internal frame

(*c*) Ignore or neglect the relativistic effects

All components or elements of the physical system are linear. The resulting mathematical model may be linear or non-linear, depending on the given physical system. Generally speaking, all physical systems exhibit non-linear behaviour. Accurate mathematical modelling of any physical system will lead to non-linear differential equations governing the behaviour of the system. Often, these non-linear differential equations have either no solution or difficult to find a solution. Assumptions are made to linearise a system, which permits quick solutions for practical purposes. The advantages of linear models are the following:

(1) their response is proportional to input

(2) superposition is applicable

(3) they closely approximate the behaviour of many dynamic systems

(4) their response characteristics can be obtained from the form of system equations without a detailed solution

(5) a closed-form solution is often possible

(6) numerical analysis techniques are well developed, and

(7) they serve as a basis for understanding more complex non-linear system behaviours.

It should, however, be noted that in most non-linear problems it is not possible to obtain closed-form analytic solutions for the equations of motion. Therefore, a computer simulation is often used for the response analysis.

When analysing the results obtained from the mathematical model, one should realise that the mathematical model is only an approximation to the true or real physical system and therefore the actual behaviour of the system may be different.

2. Formulation of governing equations

Once the mathematical model is developed, we can apply the basic laws of nature and the principles of dynamics and obtain the differential equations that govern the behaviour of the system. A basic law of nature is a physical law that is applicable to all physical systems irrespective of the material from which the system is constructed. Different materials behave differently under different operating conditions. Constitutive equations provide information about the materials of which a system is made. Application of geometric constraints such as the kinematic relationship between displacement, velocity, and acceleration is often necessary to complete the mathematical modelling of the physical system. The application of geometric constraints is necessary in order to formulate the required boundary and/or initial conditions.

The resulting mathematical model may be linear or non-linear, depending upon the behaviour of the elements or components of the dynamic system.

3. Mathematical solution of the governing equations

The mathematical modelling of a physical vibrating system results in the formulation of the governing equations of motion. Mathematical modelling of typical systems leads to a system of differential equations of motion. The governing equations of motion of a system are solved to find the response of the system. There are many techniques available for finding the solution, namely, the standard methods for the solution of ordinary differential equations, Laplace transformation methods, matrix methods, and numerical methods. In general, exact analytical solutions are available for many linear dynamic systems, but for only a few non-linear systems. Of course, exact analytical solutions are always preferable to numerical or approximate solutions.

4. Physical interpretation of the results

The solution of the governing equations of motion for the physical system generally gives the performance. To verify the validity of the model, the predicted performance is compared with the experimental results. The model may have to be refined or a new model is developed and a new prediction compared with the experimental results. Physical interpretation of the results is an important and final step in the analysis procedure. In some situations, this may involve (*a*) drawing general inferences from the mathematical solution, (*b*) development of design curves, (*c*) arrive at a simple arithmetic to arrive at a conclusion (for a typical or specific problem), and (*d*) recommendations regarding the significance of the results and any changes (if any) required or desirable in the system involved.

1.5.1 COMPONENTS OF VIBRATING SYSTEMS

(*a*) **Stiffness elements**

Some times it requires finding out the equivalent spring stiffness values when a continuous system is attached to a discrete system or when there are a number of spring elements in the system. Stiffness of continuous elastic elements such as rods, beams, and shafts, which produce restoring elastic forces, is obtained from deflection considerations.

The stiffness coefficient of the rod (Fig. 1.3) is given by $k = \dfrac{EA}{l}$

The cantilever beam (Fig.1.4) stiffness is $k = \dfrac{3EI}{l^3}$

The torsional stiffness of the shaft (Fig.1.5) is $K = \dfrac{GJ}{l}$

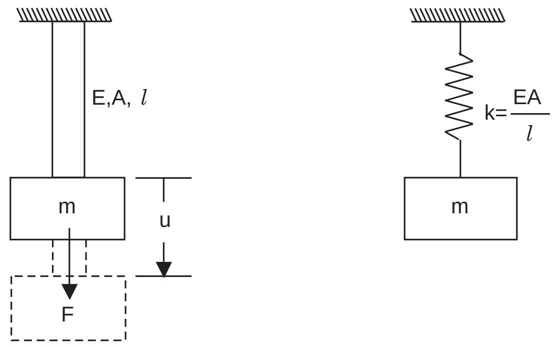

Fig.1.3 Longitudinal vibration of rods.

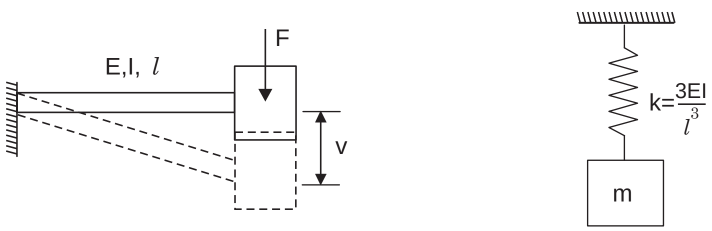

Fig.1.4 Transverse vibration of cantilever beams.

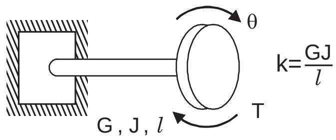

Fig. 1.5 Torsional system.

When there are several springs arranged in parallel as shown in Fig. 1.6, the equivalent spring constant is given by algebraic sum of the stiffness of individual springs. Mathematically,

$$k_{eq} = \sum_{i=1}^{n} k_i$$

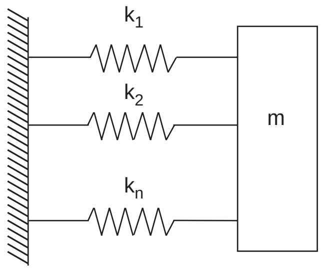

Fig. 1.6 Springs in parallel.

When the springs are arranged in series as shown in Fig. 1.7, the same force is developed in each spring and is equal to the force acting on the mass.

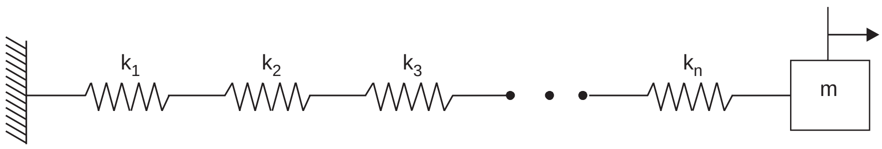

Fig. 1.7 Springs in series.

The equivalent stiffness k_{eq} is given by:

$$1/k_{eq} = \frac{1}{\sum_{i=1}^{n} \frac{1}{k_i}}$$

Hence, when elastic elements are in series, the reciprocal of the equivalent elastic constant is equal to the reciprocals of the elastic constants of the elements in the original system.

(*b*) **Mass or inertia elements**

The mass or inertia element is assumed to be a rigid body. Once the mathematical model of the physical vibrating system is developed, the mass or inertia elements of the system can be easily identified.

(*c*) **Damping elements**

In real mechanical systems, there is always energy dissipation in one form or another. The process of energy dissipation is referred to in the study of vibration as *damping*. A damper is considered to have neither mass nor elasticity. The three main forms of damping are *viscous damping*, *Coulomb* or *dry-friction damping*, and *hysteresis damping*. The most common type of energy-dissipating element used in vibrations study is the *viscous damper*, which is also referred to as a *dashpot*. In viscous damping, the damping force is proportional to the velocity of the body. Coulomb or dry-friction damping occurs when sliding contact that exists between surfaces in contact are dry or have insufficient lubrication. In this case, the damping force is constant in magnitude but opposite in direction to that of the motion. In dry-friction damping energy is dissipated as heat.

Solid materials are not perfectly elastic and when they are deformed, energy is absorbed and dissipated by the material. The effect is due to the internal friction due to the relative motion between the internal planes of the material during the deformation process. Such materials are known as visco-elastic solids and the type of damping which they exhibit is called as *structural* or *hysteretic damping*, or *material* or *solid damping*.

In many practical applications, several dashpots are used in combination. It is quite possible to replace these combinations of dashpots by a single dashpot of an equivalent damping coefficient so that the behaviour of the system with the equivalent dashpot is considered identical to the behaviour of the actual system.

1.6 FREE VIBRATION OF SINGLE DEGREE OF FREEDOM SYSTEMS

The most basic mechanical system is the *single-degree-of-freedom system*, which is characterized by the fact that its motion is described by a single variable or coordinates. Such a model is often used as an approximation for a generally more complex system. Excitations can be broadly divided into two types, initial excitations and externally applied forces. The behavior of a system characterized by the motion caused by these excitations is called as the *system response*. The motion is generally described by displacements.

1.6.1 FREE VIBRATION OF AN UNDAMPED TRANSLATIONAL SYSTEM

The simplest model of a vibrating mechanical system consists of a single mass element which is connected to a rigid support through a linearly elastic massless spring as shown in Fig. 1.8. The mass is constrained to move only in the vertical direction. The motion of the system is described by a single coordinate $x(t)$ and hence it has one degree of freedom (DOF).

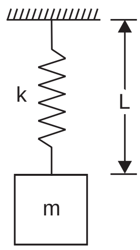

Fig. 1.8 Spring mass system.

The equation of motion for the free vibration of an undamped single degree of freedom system can be rewritten as

$$m\ddot{x}(t) + kx\,(t) = 0$$

Dividing through by m, the equation can be written in the form

$$\ddot{x}(t) + \omega_n^2 x\,(t) = 0$$

in which $\omega_n = \sqrt{k/m}$ is a real constant. The solution of this equation is obtained from the initial conditions

$$x(0) = x_0,\ \dot{x}(0) = v_0$$

where x_0 and v_0 are the initial displacement and initial velocity, respectively.

The general solution can be written as

$$x(t) = A_1 e^{i\omega_n t} + A_2 e^{-i\omega_n t}$$

in which A_1 and A_2 are constants of integration, both complex quantities. It can be finally simplified as:

$$x(t) = \frac{X}{2}\left[e^{i(\omega_n t - \phi)} + e^{-i(\omega_n t - \phi)}\right] = X \cos\,(\omega_n t - \phi)$$

so that now the constants of integration are X and ϕ.

This equation represents harmonic oscillation, for which reason such a system is called a *harmonic oscillator*.

There are three quantities defining the response, the *amplitude* X, the *phase angle* ϕ and the *frequency* ω_n, the first two depending on external factors, namely, the initial excitations, and the third depending on internal factors, namely, the system parameters. On the other hand, for a given system, the frequency of the response is a characteristic of the system that stays always the same, independently of the initial excitations. For this reason, ω_n is called the *natural frequency* of the harmonic oscillator.

The constants X and ϕ are obtained from the initial conditions of the system as follows:

$$X = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}$$

and

$$\phi = \tan^{-1}\left[\frac{v_0}{x_0\omega_n}\right]$$

The *time period* τ, is defined as the time necessary for the system to complete one vibration cycle, or as the time between two consecutive peaks. It is related to the natural frequency by

$$\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{k}}$$

Note that the natural frequency can also be defined as the reciprocal of the period, or

$$f_n = \frac{1}{\tau} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

in which case it has units of cycles per second (cps), where one cycle per second is known as one Hertz (Hz).

1.6.2 FREE VIBRATION OF AN UNDAMPED TORSIONAL SYSTEM

A mass attached to the end of the shaft is a simple torsional system (Fig. 1.9). The mass of the shaft is considered to be small in comparison to the mass of the disk and is therefore neglected.

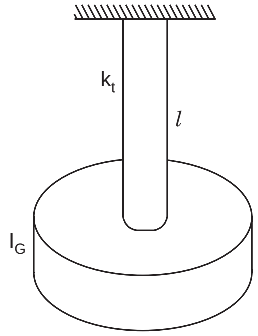

Fig. 1.9 Torsional system.

The torque that produces the twist M_t is given by

$$M_t = \frac{GJ}{l}$$

where J = the polar mass moment of inertia of the shaft $\left(J = \frac{\pi d^4}{32} \right.$ for a circular shaft of diameter $\left. d \right)$

G = shear modulus of the material of the shaft.

l = length of the shaft.

The torsional spring constant k_t is defined as

$$k_t = \frac{T}{\theta} = \frac{GJ}{l}$$

The equation of motion of the system can be written as:

$$I_G\ddot{\theta} + k_t\theta = 0$$

The natural circular frequency of such a torsional system is $\omega_n = \left(\frac{k_t}{I_G} \right)^{1/2}$

The general solution of equation of motion is given by

$$\theta(t) = \theta_0 \cos \omega_n t + \frac{\dot{\theta}_0}{\omega_n} \sin \omega_n t$$

1.6.3 ENERGY METHOD

Free vibration of systems involves the cyclic interchange of kinetic and potential energy. In undamped free vibrating systems, no energy is dissipated or removed from the system. The kinetic energy T is stored in the mass by virtue of its velocity and the potential energy U is stored in the form of strain energy in elastic deformation. Since the total energy in the system is constant, the principle of conservation of mechanical energy applies. Since the mechanical energy is conserved, the sum of the kinetic energy and potential energy is constant and its rate of change is zero. This principle can be expressed as

$$T + U = \text{constant}$$

or $$\frac{d}{dt}(T + U) = 0$$

where T and U denote the kinetic and potential energy, respectively. The principle of conservation of energy can be restated by

$$T_1 + U_1 = T_2 + U_2$$

where the subscripts 1 and 2 denote two different instances of time when the mass is passing through its static equilibrium position and select $U_1 = 0$ as reference for the potential energy. Subscript 2 indicates the time corresponding to the maximum displacement of the mass at this position, we have then

$$T_2 = 0$$

and $$T_1 + 0 = 0 + U_2$$

If the system is undergoing harmonic motion, then T_1 and U_2 denote the maximum values of T and U, respectively and therefore last equation becomes

$$T_{\max} = U_{\max}$$

It is quite useful in calculating the natural frequency directly.

1.6.4 STABILITY OF UNDAMPED LINEAR SYSTEMS

The mass/inertia and stiffness parameters have an affect on the stability of an undamped single degree of freedom vibratory system. The mass and stiffness coefficients enter into the characteristic equation which defines the response of the system. Hence, any changes in these coefficient will lead to changes in the system behavior or response. In this section, the effects of the system inertia and stiffness parameters on the stability of the motion of an undamped single degree of freedom system are examined. It can be shown that by a proper selection of the inertia and stiffness coefficients, the instability of the motion of the system can be avoided. A stable system is one which executes bounded oscillations about the equilibrium position.

1.6.5 FREE VIBRATION WITH VISCOUS DAMPING

Viscous damping force is proportional to the velocity $\dot{x}$ of the mass and acting in the direction opposite to the velocity of the mass and can be expressed as

$$F = c\dot{x}$$

where c is the damping constant or coefficient of viscous damping. The differential equation of motion for free vibration of a damped spring-mass system (Fig. 1.10) is written as:

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

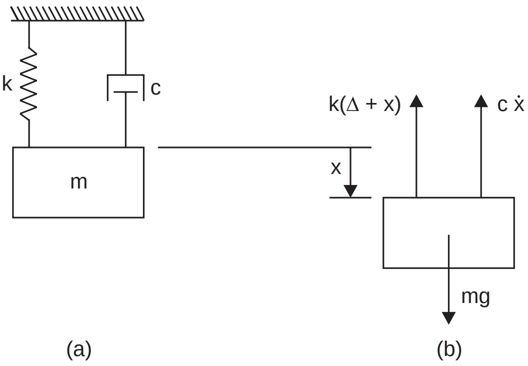

Fig. 1.10 Damped spring-mass system.

By assuming $x(t) = Ce^{st}$ as the solution, the auxiliary equation obtained is

$$s^2 + \frac{c}{m}s + \frac{k}{m} = 0$$

which has the roots

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

The solution takes one of three forms, depending on whether the quantity $(c/2m)^2 - k/m$ is zero, positive, or negative. If this quantity is zero,

$$c = 2m\omega_n$$

This results in repeated roots $s_1 = s_2 = -c/2m$, and the solution is

$$x(t) = (A + Bt)e^{-(c/2m)t}$$

As the case in which repeated roots occur has special significance, we shall refer to the corresponding value of the damping constant as the *critical damping constant*, denoted by $C_c = 2m\omega_n$. The roots can be written as:

$$s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$$

where $\omega_n = (k/m)^{1/2}$ is the circular frequency of the corresponding undamped system, and

$$\zeta = \frac{c}{C_c} = \frac{c}{2m\omega_n}$$

is known as the *damping factor*.

If $\zeta < 1$, the roots are both imaginary and the solution for the motion is

$$x(t) = Xe^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

where $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$ is called the damped circular frequency which is always less than ω, and ϕ is the phase angle of the damped oscillations. The general form of the motion is shown in Fig. 1.11. For motion of this type, the system is said to be *underdamped.*

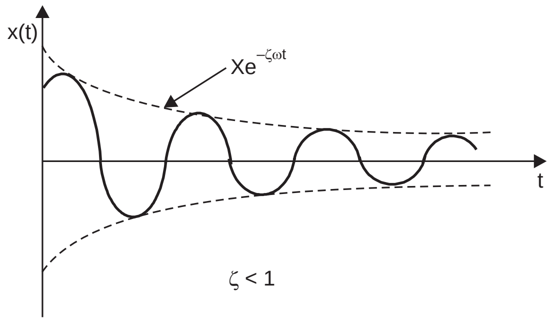

Fig. 1.11 The general form of motion.

If $\zeta = 1$, the damping constant is equal to the critical damping constant, and the system is said to be *critically damped*. The displacement is given by

$$x(t) = (A + Bt)e^{-\omega_n t}$$

The solution is the product of a linear function of time and a decaying exponential. Depending on the values of A and B, many forms of motion are possible, but each form is characterized by amplitude which decays without oscillations, such as is shown in Fig. 1.12.

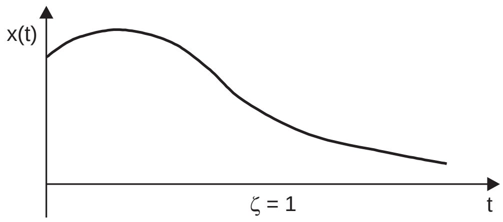

Fig. 1.12 Amplitude decaying without oscillations.

In this case $\zeta > 1$, and the system is said to be *overdamped*. The solution is given by:

$$x(t) = C_1 e^{(-\zeta + \sqrt{\zeta^2 - 1})\,\omega_n t} + C_2 e^{(-\zeta - \sqrt{\zeta^2 - 1})\,\omega_n t}$$

The motion will be non-oscillatory and will be similar to that shown in Fig. 1.13.

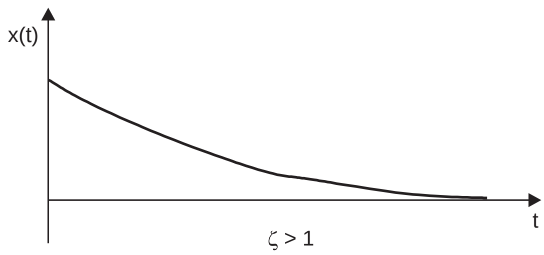

Fig. 1.13 Non-oscillatory motion.

1.6.6 LOGARITHMIC DECREMENT

The logarithmic decrement represents the rate at which the amplitude of a free damped vibration decreases. It is defined as the natural logarithm of the ratio of any two successive amplitudes.